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Angular momentum between physics and mathematics

1. Introduction

Angular momentum is one of the fundamental notions of modern physics. It can be defined in classical mechanics, electromagnetism, quantum mechanics and quantum field theory and, although the mathematical expressions and observable phenomena linked to it are in each case different, the conservation of angular momentum is regarded as holding for any system which is invariant under rotation. It is not my intention to discuss here the differences between the various notions of angular momentum, but rather to underscore how, despite those differences, that concept today maintains a strong identity as the "same" physical quantity. To quote a view from the scientific community:

> "The concept of angular momentum, defined initially as the moment of momentum ($L = r \times p$), originated very early in classical mechanics (Kepler's second law, in fact, contains precisely this concept.) Nevertheless, angular momentum had, for the development of classical mechanics, nothing like the central role this concept enjoyes in quantum physics. Wigner[1] notes, for example that most books on mechanics written around the turn of the century (and even later) do not mention the general theorem of the conservation of angular momentum. In fact, Cajori's well-known "History of

[1] Wigner 1967, p. 14.

> physics"[2] (1929 edition) gives exactly half a line to angular momentum conservation. That the concept of angular momentum may be of greater importance in quantum mechanics is almost self-evident. The Planck quantum of action has precisely the dimensions of an angular momentum, and, moreover, the Bohr quantisation hypothesis specified the unit of (orbital) angular momentum to be $h/2\pi$. Angular momentum and quantum physics are thus clearly linked.“[3]

In this passage angular momentum is presented as a physical entity with a classical and a quantum incarnation. This situation is not peculiar to that notion, and there are a number of classical mechanical concepts which have been taken over into quantum theory without losing connection to their classical selves. I believe this to be a very important aspect of the relationship between mathematics and physics and in particular of the complex nature of physical-mathematical notions. Historically, such concepts do not appear because a physical content meets a mathematical form, but rather emerge from a coevolution of mathematics and physics making evident both the multiplicity within each discipline and the close correlation - at times even indistinguishability - between specific aspects of physical and mathematical practice, as well as of the philosophical and technological contexts in which they are embedded. It is because of this complex, composite character that physical-mathematical notions can be perceived by scientists as possessing a specific identity behind the many representation they can be encountered in - from Kepler’s area law to the quantum numbers of the Bohr-Sommerfeld atom. In the following pages, I shall tentatively explore this constellation by sketching the emergence of classical angular momentum and its translation into quantum-theoretical terms.

2. Johannes Kepler's area law and Isaac Newton's parallelogram of forces

Other than linear motion, rotations have attracted the attention of mathematically-minded philosophers since Antiquity. Although this was largely due to the evident regularities and outstanding cultural significance of heavenly motion, one must not forget that the stability of rotating bodies could also be inferred from everyday experience and was at the basis of simple tools such as the potter's wheel or the spinning top, whose use is attested well before the emergence of geometrical or numerical representations of celestial motion.[4] The practice of discus-throwing presupposed a highly refined understanding of the rotation of rigid bodies and flywheels were employed already in Antiquitiy to stabilize the motion of machines of various kind.[5] Thus, it is not suprising that in pre-modern natural philosophical systems, especially but not only the Aristotelian one, circular motion had a special status as a "perfect" movement which partained to celestial entities.[6] The geometrical models of celestial motion based on circles were the starting point for the development of modern mechanics and Newtonian gravitation - a development which ironically led to the rejection of the idea of the perfection of rotation in favour of a higher consideration of linear movement. While Nicholaus Copernicus had still adhered to the notion that celestial movements had a circular form, Johannes Kepler expressed them by means of ellipses.[7] In his model, the stability of the Ptolemaic spherical cosmos found a new expression in the statement that the elliptical orbits of the planets were fixed both in shape and space orientation. Moreover, the movement of celestial bodies along their path was such, that the areas spanned by the line connecting a planet to the Sun were proportional to the time elapsed, despite the fact that the distance between the two bodies and the velocity of the planet constantly changed. As we shall see, the habit of expressing the constancy of rotational motion in terms of areas will remain alive until the 19th century, so that what is today referred to as the conservation of angular momentum at that

[2] Cajori 1929.
[3] Biedenharn, Louck and Carruthers 1981, p. 1.
[4] Hurschmann 1999; Scheibler 1999.
[5] Decker 1997; Krafft 1999, esp. col. 1087.
[6] Daxelmüller 1999.
[7] Dugas 1988, p. 110-119, Kepler 1628, p. 410-412.

time took the form of a principle of conservation of areas.
Before proceeding in our exploration of the methods employed in the early modern period to formalize and analyse rotations, we have to make a clear distinction between the graphic representation of mechanical and dynamical quantitites, their analytical expressions and the abstract mathematical structure which are associated with them today.[8] The angular momentum of a classical mechanical system is mathematically represented today by an axial vector in three-dimensional space, which can be manipulated according to the rules of vector algebra and is graphically depicted as an oriented segment in space. Vector algebra was only developed from the middle of the 19th century onward and played no role in the emergence of classical mechanics, but the representation and manipulations of some physical quantities (motion, force) by means of oriented segments was current already in the 17th century.
The composition of forces with the parallelogram rule had been in use since the Renaissance and was further developed by Isaac Newton.[9] To compose the effect of two forces acting on the same body, Newton represented them by two segments, each with length and direction corresponding to the motion which the force would impart on the body by acting on it for a given time.[10] The segments were drawn as the sides of a parallelogram whose diagonal represented the combined effect of the two forces. In this procedure force was represented and manipulated geometrically as the motion it could impart to a body and this was in turn connected to an idea of force which Newton had taken over from medieval tradition. It is not here the place to discuss Newton's complex and at times ambiguous idea of force: suffice to say that, while innovative, it still embedded the earlier concept of a discrete "impetus" which, when transmitted to a body, set it into a motion of direction and extension corresponding to its own entity.[11]
Although Newton employed a geometrical representation of forces and motions, he never used it for angular momentum, for the very simple reason that no such notion can be found in his work - not even where he discussed the problem of the precession of the Earth's axis.[12] According to the analysis of Clifford Truesdell, the first author to speak not only of a "moment of rotational motion", but also of its "conservation" ("conservationem momentii motus rotatorii") was Daniel Bernoulli, who did so in a letter written in February 1744.[13] Bernoulli had discussed the motion of a ball sliding within a rotating tube, demonstrating that what we regard as the absolute value of the angular mometnum of the whole system could not be changed by the mutual interaction of its parts. By referring to these results as a conservation of “moment of rotational motion”, he was using an expression, the “moment” of a force, which had been developed in the context of the theory of the lever. The effect of a force of intensity I acting on a lever is proportional both to I and to the distance L of its point of application from the fulcrum. The "moment" of that force acting in that specific configuration is equal to the product IL and gives a scalar measure of the effect of the force. In the late Renaissance this notion was extended to indicate the effect of a force acting not only on a lever, but on a generic body of which a point remained fixed (e.g. a pendulum).[14] Daniel Bernoulli extended it further, but still regarded the moment of rotational motion as a scalar quantity and did not associate any direction to it.

3. Leonard Euler on the rotation of rigid bodies

While Kepler and Newton had mainly dealt with systems of mass points interacting with each other, mathematicians of the 18th century took up the task of mathematizing the motions of extended

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[8] This is a very complex subject that has been extensively treated in the historical literature (Caparrini 1999, 2002; Crowe 1985) and I will only deal with it as far as necessary for the present investigation.
[9] Dugas 1988, p. 123-127, 151-153, 207-209.
[10] Dugas 1988, p. 208-209, Kutschmann 1983, p. 126-127.
[11] Kutschmann 1983, p. 18-19, 120-129.
[12] Dobson 1998, especially p. 132-133, 136-138. Truesdell 1964b, p. 244-245.
[13] Truesdell 1964b, p. 254-256, quote from Bernoulli 1744, p. 549.
[14] Truesdell 1964b, p. 248-252.

bodies on which forces could be applied at the same time at different places. Decisive contributions to this field were given by Leonard Euler, who was the first to write down the general equations of motion for an extended body.[15] Starting from the recognition that any infinitesimal motion of a body can be decomposed into a translation and a rotation, Euler developed in a series of papers the mathematical analysis of the movement of rigid bodies and wrote down the differential equations governing it. In his writing he offered different derivations of his results, and I shall focus on the latest one (1775), which was also the most accomplished. To express mathematically the state of a body Euler introduced the three angles which today still bear his name, and thanks to which a parametrisation of any rotational motion is possible.[16] These new quantities allowed him to transform a geometrical description given in terms of axes of rotation and space positions into an analytical one based on trigonometric functions. This was a very important step, because it allowed Euler and later authors to at least partly discard the geometrical language of rotation in favour of the purely algebrical ("analytical") one. It is not necessary for us to follow Euler's derivation and it will suffice to state the equations as he wrote them in 1775:

$$\int dM(ddx/dt^2) = iP$$
$$\int dM(ddy/dt^2) = iQ$$
$$\int dM(ddz/dt^2) = iR$$
$$\int zdM(ddy/dt^2) - \int ydM(ddz/dt^2) = iS$$
$$\int xdM(ddz/dt^2) - \int zdM(ddx/dt^2) = iT$$
$$\int ydM(ddx/dt^2) - \int xdM(ddy/dt^2) = iU$$ [17]

In these formulas dM represents an infinitesimal mass element of the body at the position with Cartesian coordinates (x,y,z); ddx/dt^2 (i.e. d^2x/dt^2) etc. are the corresponding accelerations; P, Q and R are the resultant external forces acting in the directions of the three axes x, y and z; S, T and U are the resultant “moments” of the external forces, again taken in the directions x, y and z.
Euler used here the notion of "moment" like Daniel Bernoulli had done, i.e. in a scalar sense, and so did not regard S, T, and U as components of a single physical entity, but rather as three separate moments computed with respect to the three axes. Euler's first three formulas state the relationship between force, mass and acceleration, while the last three expressions formally correspond to what we today describe as the relationship between the (vectorial) moment of external force (M_x, M_y, M_z) and the time derivative of (vectorial) angular momentum (J_x, J_y, J_z), whose components are defined in the same way as in Euler's equations.[18] Therefore, from a purely analytical point of view, one may claim that Euler had written down both the expression and the dynamics of the angular momentum of a solid body. Moreover, the equations implied that, in absence of external moments of force, the value of the angular momentum would be conserved.
Howerver, Euler did not consider the equations as referring to the evolution of the three components of the same quantitiy. Indeed, he did not even seem to regard the individual expressions as particularly significant. In a later paper he discussed the fact that the effects of the moments S, T and U could indeed be composed in the same way as forces, i.e. using the rule of the parallelogram.[19] Thus, it seems that he was becoming aware that his analytical expressions could be somehow translated back into a geometrical form. However, at that time Euler was already very old and blind and therefore could not further pursue this research. The fact that the great mathematician only became aware at such a late date of this aspect of the subject which he had studied for so long is in my opinion the best evidence that such changes of perspective are anything but trivial.

[15] Blanc 1968; Caparrini 1999; Truesdell 1964a, 1964b, on which the following discussion is largely based.
[16] Euler 1775, p. 208-211, i.e. p.103-104.
[17] Euler 1775, p. 224-225, i.e. 113.
[18] Davis 2002, esp. p. 255-256.
[19] Caparrini 2002, p. 154-155.

4."Conservation of area" and "invariable plane" in French mathematics (1788-1790)

Euler's equations were later taken up by other authors, embedded in new systems of mechanics and eventually rederived according to new principles.[20] In his "Mechanique analytique" (1788) Joseph Louis Lagrange expressed them in the formalism that still carries his name and in which the "vectorial" character of the equations was less evident than in Euler's original form.[21] However, Lagrange noted that the new formalism allowed to deduce a number of principles of conservation which had hitherto been regarded separately: "the conservation of living force, the conservation of the movement of the centre of gravity, the conservation of the moment of rotation or principle of the areas and the principle of least action".[22] Lagrange went on to explain that the principle of conservation of moment of rotation (i.e. of areas) had been derived independently by Leonard Euler, Daniel Bernoulli and Patrick d'Arcy.[23] We have already seen what Euler and Bernoulli had worked on. According to Lagrange, d'Arcy had formulated a special case of this result in terms of areas: "la somme des produits de la masse de chaque corps par l'aire que son rayon vecteur décrit autour d'un centre fixe sur un même plan de projection est toujours proportionelle au temps".[24] Lagrange regarded d'Arcy's formulation as "généralisation du beau théorème de Newton", which in turn was a generalisation of Kepler's law of areas, and, when deriving the result with his own methods, he referred to it as "principle of areas".[25] Thus, by the late 18th century, the notion that a freely rotating system was subject to a specific conservation law was present, but the law was mainly regarded as concerning one or more scalar quantities. It was Pierre Simon Laplace who drew attention to the fact that the principle of areas also implied the conservation of a preferred direction of the system, and he expressed this fact geometrically in terms of an "invariable plane" of rotation, which for us corresponds to the plane perpendicular to angular momentum.[26]
As Euler had done, Laplace wrote down the expression of what we regard as the three components of angular momentum and noted that they were constant in absence of external moments of force. He also remarked, like Lagrange had done, that these quantites could be interpreted in terms of areas and that one could choose the coordinate system in such a way that two of the constant quantities would be zero, while the third one had the highest possible value of any of them. It is easy to interpret this result by conceiving of the three quantities as components of a vector, but Laplace chose to adhere to the "area" interpretation. This may appear somehow forced to a modern reader, but for someone like Laplace who had been working many years on celestial mechanics the connection between his new result and Kepler's law probably appeared rather intuitive, while the notion of associating an oriented segment to some rather abstract analytical expression did not. It would be incorrect to say that Laplace rejected geometrical interpretations of his analytical formulas: he only chose a different one that we do today. As we shall see in the next section, the first one to propose a geometrical interpretation similar to the modern one was the French mathematician Louis Poinsot.

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[20] Grattan-Guinness 1990, p. 270-301.
[21] Truesdell 1964b, p. 245-246.
[22] "théorème connus sous les noms de conservation des forces vives, conservation du mouvement du centre de gravité, de conservation des moments de rotation ou principe des aires, et de principe de la moindre quantité d'sction" Lagrange 1853, p. 257. I quote from a later edition of Lagrange's work, which however does not present relevant difference to the first one as far as our subject is concerned.
[23] Lagrange 1853, p. 259-261.
[24] Lagrange 1853, p. 260.
[25] Lagrange 1853, p. 260, 278-288.
[26] Laplace, 1799, p. 65-69. Laplace’s work is discussed by Caparrini 2002, p. 156-157; Grattan-Guinness 1990, p. 317-318, 360. Grattan-Guinness writes that Laplace had “in effect” shown some properties of angular momentum – it is important to note that Laplace made no use of such notion.

5. Louis Poinsot's statics and the notion of a couple (1803)

Louis Poinsot had set out to become an engineer first at the École Politechnique and then at the École des Ponts et Chaussés, but he eventually gave up his study to pursue his interest in mathematics and in 1804 became a teacher of that discipline at the Lycée Bonaparte.[27] In 1803 he published a "Treatise on Statics" which, although written for candidates to the École Polytechnique, was much appreciated by all engineers and also by some French academics.[28] Thanks to that work and to a series of memoires on rotational motion, in 1809 he obtained the post of inspector general at the University and in 1813 was elected to the Academy. He remained active in research and teaching at the university and the École Poythechnique, but was often in opposition to the analytical school of mathematics because of his geometrical approach to mechanics. In the course of the 19th century his work found increasing appreciation among French mathematicians. In 1858 Joseph Louis François Bertrand stated in a discourse:[29]

> Nul oserait [...] aujourd'hui contester l'importance et la hauteur des travaux mécaniques de Poinsot: il semble évident déja que la postérité doit placer l'illustre auteur de la 'Statique' bien au-dessus des contemporaines, jadis plus célèbre, qui l'ont si longtemps méconnu. Poisson disait, au sein même, je crois, du Bureau des longitudes: 'si Poinsot se présentait á l'École polytechnique, ma conscience ne me permetterait pas de l'y admettre'[30]

Poinsot's "Treatise on Statics", which reached its 12th edition in 1877, almost twenty years after the death of its author, offered a formulation of classical mechanics relying on geometrical representations, as advocated by Gaspard Monge of whom Poinsot was a follower. However, Poinsot not only gave a different presentation to old material, but also used the new form to develop innovative and heuristically fruitful physical mathematical notions.

At the centre of the book stood the concept of a couple, i.e. a system of two equal and opposite forces acting on two points of the same body. The effect of a couple could never be reduced to that of a single force, as it corresponded to a rotation around an axis perpendicular to the plane of the two forces.[31] The intensity of the effect of a couple was measured by the (scalar) moment of the couple (i.e. intensity of the forces times their distance) and Poinsot proposed to represent that moment geometrically, by means of an oriented segment perpendicular to the plane of the couple.[32] Poinsot showed how, thanks to this representation, the effect of two couples could be composed by using the rule of the parallelogram, exactly as in the case of forces. Using the notion of a couple Poinsot showed that the total effect of a system of forces on a body could always be represented as the combination of a single resultant force and a single resultant couple. We do not need to go further into his theory, but it is important to stress that, despite its geometrical form, it was by no means "intuitive" in the sense that it appealed to some notions immediately linked to everyday experience, as in the case of force and linear motion. While the representation of forces by means of oriented segments was immediately suggested by the motion they impressed, no such obvious interpretation existed for couples and rotations. As we have seen, momenta were usually conceived as scalar quantities. Like Laplace's "principle of areas" and "invariable plane", Poinsot's theory was the translation into geometrical forms of a complex, abstract notion that had been developed by analytical means. Neither of the alternative "geometrisations" of the dynamics of rotating bodies was more immediate and intuitive than the other: they were simply linked to different physical systems which the authors had in mind, on the one side the Solar system, on the other the spinning top. Poinsot's theory proved immediately succesful with engineers, who were capable of dealing well with geometrical entities, while Laplace's method was more appreciated by mathematicians.

[27] On Poinsot's life and work see: Grattan-Guinness 1990, p. 190-191, 358-364, 1154-1157, 1233-1236; Taton 1975.
[28] Poinsot 1803.
[29] Bertrand 1858. For the context of the text see: Tobin 2003, p. 242-244.
[30] Bertrand 1858, p. vii-viii.
[31] Poinsot 1803, p. 47.
[32] Poinsot 1803, p. 58-59.

6. Louis Poinsot's dynamics and the "conservation of forces and moments" (1806)

As befit its subject, the treatise on statics only dealt with bodies in equilibrium, but already in 1806 Poinsot started applying his approach to dynamics. In a memoire presented to the Academy he summarized his theory of couples, stressing how the geometrical representation of the moments of a couple could be used to represent and manipulate the moments of any force.[33] He showed how his method allowed to reproduce all results present in Laplace's mechanics and finally claimed that, thanks to the new formalism, "hidden forces" had emerged: "Que ces sortes de produits qu'on appelle momens n'étaient au fond que la mesure de certaines forces cachées que les couples ont mises en évidence."[34]

The meaning of this statement became somehow clearer in the third part of the essay, where the theory was applied to dynamics.[35] When a body moves freely in space in a straight line, said Poinsot, the "force" animating it remains constant in intensity and direction, and the same applies to to its moment. This "conservation of forces" and "conservation of moments" was valid for any system of bodies interacting only with each other. At this point, the term "force" was used in a slightly different meaning than in the treatment of statics, but Poinsot did not elaborate on this and offered a purely verbal "raisonnement" to prove the conservation.[36] The reasonement was based on the idea that, in each mutual interaction, the elements of the system only exchanged forces and moments with each other, so that the sum remained constant:

> On voit donc que, dans un system de corps qui ont reçu des impulsions primitives, et qui réagissant d'une manière quelconque les unes sur les autres, la somme de toutes les forces qui les animent, estimées suivant une même droit, est la somme de leurs momens par rapport à un même axe fixe quelconque, demeurent constamment mesmêmes.[37]

Poinsot stated that this conservation corresponded to two analytical principles: the conservation of the motion of the centre of gravity and the conservation of areas. These conserved quantities were expressions of "powers" ("puissances") imparted to the bodies and conserved in them.[38] Poinsot used a notion of "force" or "power" similar the one we found in Newton and such "Newtonian" concepts were not uncommon in France: Laplace, for example, used them.[39] The novelty of Poinsot's approach was that he had extended that treatment to moments of forces and in doing so he had revealed new, "hidden forces", i.e. physical entities analogous to impulse but linked to rotational motion and capable of being represented by a directed segment. In this way, the conservation of area became the conservation of a new physical mathematical quantity. Poinsot did not regard analytical expressions as defining the quantity, but only as giving its measure.[40] To sum up, Poinsot had taken the results of the analytical investigations of rotations and transformed them into a new geometrical form which brought to light an analogy between linear motion and rotation. He interpreted this analogy as the discovery of a “hidden” physical entity whose measure was given by the moment of the “force” animating a rotating body.

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[33] Poinsot 1806.
[34] Poinsot 1806, p. 345.
[35] Poinsot 1806, p. 359-365.
[36] Poinsot 1806, p. 360-361. For a discussion of Poinsot’s proof see Caparrini 1999, p. 51-53.
[37] Poinsot 1806, p. 361.
[38] Poinsot 1806, p. 346.
[39] Dugas 1988, p. 354-360.
[40] Poinsot 1806, p. 362.

7. Reception and critique of the theory of couples. Poinsot's "New theory of rotational motion" (1834)

French mathematicians appreciated Poinsot's approach, but not his geometrical formalism or his physical interpretation, and tried to give alternative analytical formulations of his results. Silvio Caparrini has given a thorough account of how, in studying rotation, scholars started developing an analytical formalism which in many ways corresponded to vector algebra, and I shall only sum up his remarks, which offer a clear example of coevolution of physics and mathematics.[41] Simeon Denis Poisson hardly mentioned couples and gave no importance to the notion of "momentum", Jacques Philippe Marie Binet introduced the notion of "aeorial velocities", Jacques Frédéric Français developed an analytical theory employing Poinsot's idea of the conservation of couples and mentioned the conservation of moments of rotation, but only in terms of the three components.[42] Thus, while Poinsot's results were slowly embedded in the analytical context, his idea of a new physical mathematical notion found little attention.

In 1826 Augustine Louis Cauchy published a series of essays on his new theory of "momens lineares", in which he reformulated and partly generalized Poinsot's geometrical theory of moments of force.[43] Cauchy showed hot to construct the "vectorial" moment of any quantity represented by a directed segment and mentioned the quantity of motion as an example, although he only treated exensively the case of moments of forces.[44] Poinsot accused him of having simply translated his own theory of couples and moments into another form and a dispute ensued in whose course Poisson defended Cauchy by claiming that Poinsot's result had already been obtained by Euler and Laplace. Poinsot replied to this accusation by underscoring the importance of giving physical content to analytical expressions. He summed up the results by Euler and Laplace and then stated:

> Mais il faut bien remarquer ici que ces théorèmes ne constituent point la composition proprement dite des moments. Cette composition n'a été, et je diraei même, n'a pu être connue que par la théorie des couples. Et en effet, ce qu'on appelait le moment d'une force par rapport à un point, ou un axe fixe, n'était jusque-là, pour les géomètre, qu'une simple expression de calcul, un produit abstrait de deux nombres, dont l'un marque une certain force, et l'autre une certaine ligne; et il me semble qu'il ne pouvait venir à personne l'idée de chercher des lois de composition, c'est-à-dire, des lois d'équilibre entre de tels produits. [...] il fallait une notion statique, qui manquait alors aux géomètres, et cette notion est celle du couple.[45]

Poinsot was here of course arguing "pro domo sua", but the best proof that his geometrical physical interpretation of previous analytical results was an original, fruitful contribution to the science of mechanics was the fact that, thanks to it, he could bring forward a "New theory of the rotation of bodies" ("Théorie nouvelle de la rotation des corps", 1834) for which he is mostly remembered today. In 1834 Poinsot presented his work to the Paris Academy and then published it as a short memoire in which he only made use of geometrical arguments expressed in verbal form: no analytical formulas were present.[46] In this text he employed his methods of geometrical representation to express the motion of a freely rotating body in terms of two cones along which the instantaneous axis of rotation of the body moved. Almost twenty years later, in 1851, he published a book with the same title of the memoire in which the previous results were expressed also in analytical form and expanded upon.[47] In this later text Poinsot took up again the subject of conservation of forces and moments, which he here referred to as "conservation of forces and of couples".[48]

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[41] Caparrini 2002.
[42] Caparrini 2002, p. 160-162, 167-170; Grattan-Guinness 1990, p. 364-365, 368-370; Français 1813, p. 21-23.
[43] Caprrini 2002, p. 171-172, Grattan-Guinness 1990, p. 1154-1157.
[44] Cauchy 1826.
[45] Poinsot 1827, p. 4-5.
[46] Grattan-Guinness 1990, p. 1233-1235; Poinsot 1834a.
[47] Poinsot 1951b.
[48] Poinsot 1851b, p. 45-49.

French mathematicians once again showed more interest in translating Poinsot's theory into analytical terms than in further developing his geometrical approach and his ideas of new conserved "forces" associated to rotations. However, the new theory of rotational motion was appreciated by engineers and won special praise from Léon Foucault, best known for his demonstration of the rotation of the Earth by means of a pendulum.[49] Two of Foucault's devices - the pendulum and the gyroscope - play a very important role in our story and I shall discuss them in the next section.

8. Foucault's pendulum, his gyroscope and the English reception of Poinsot's theory (1851-1855)

Jean Bernard Léon Foucault, self-taught natural philosopher and inventor, had achieved his first natural philosophical recognition thanks to experiments on the velocity of light.[50] Around 1850 he conceived the idea of building a large pendulum whose plane of oscillation would slowly change in orientation with respect to a terrrestial observer because of the rotation of the Earth. Foucault experimented at first in his own basement, but was then allowed to set up his pendulum at the Paris Observatory and in February 1851 presented his results to the Academy: the measured daily deviation of the oscillation plane from the terrestrial vertical was given by a simple formula in which the sine of the angle expressing the local latitude appeared. Foucault's result were greeted with interest and the experiment was repeated in the Paris Pantheon for the broader public: the experiment was an instant success and was soon replicated both in France and abroad. A pendulum was swinging in London already in early April, a few months later also in many other British towns.

However, Foucault's pendulum was much more than a popular demonstration in which a scientific theory could be shown to correspond to experience: while the motion of the pendulum did indeed represent well-established astronomical and mechanical knowledge, it did so in a particulary simple form which not only was immediately evident to the eye (as long as the pendulum was long enough), but could also be expressed in a very elementary mathematical form, i.e. a sinus factor. Yet the analytical theories of rotations showed none of that simplicity and French mathematicians felt challenged to relate the simplicity of the pendulum to the complexity of the formulas. In other words, a tension between two different representations of the laws of rotation - the pendulum and the equations - had been constructed and now had to be resolved, possibly without declaring either the equations or the pendulum as wrong. As we shall see, this was possible thanks to Poinsot's theory of rotations.

In the short memoire discussing his experiments, Foucault had only offered a very sketchy argument to justify the sine factor: a pendulum at the Pole would have an oscillating plane which remained constant while the earth rotated under it, and thus would appear to a terrestrial observer a making a complete 360° rotation each day.[51] However, a pendulum standing at a generic latitude would be forced to rotate along with the earth, and thus would have a more complex motion, which Foucault regarded as a problem for mathematicians to solve: "Mais quand on descend vers nos latitude, le phénomene se complique d'un élément assez difficile à apprécier et sur le quel je souhaite bien vivement l'attention des géometres."[52] He claimed to have performed an approximate computation leading to the prediction of the sinus factor which the experiment confirmed. A few days later Jacques Binet, who as we saw had written a treatise on rotational motion, published a short note in which he, as a representative of the "géometres", rose to the challenge posed by Foucault.[53] He described Foucault's results as "unexpected" ("inattendu"), and continued: "En me consultant, l'auteur [i.e. Foucault] désirait savoir à quel point le résultant mécanique auquel il

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[49] Tobin 2003, here especially p. 150-151, 161.

[50] The following discussion of Foucault's pendulum and gyroscope is based on: Broelmann 2002, p. 42- 50, Tobin 2003, p. 137-160.

[51] Foucault 1851.

[52] Foucault 1851, p. 136.

[53] Binet 1851.

arrivait s'accordait avec la theorie mathématique et avec les déductions obtenues par les géomètres."[54] Binet explained that Laplace had devoted some attention to similar subjects, but without deriving any relevant results and that: "Poisson á traité ce sujet [...]; cependant ce nétait pas l'object spécial de ce grand géometre, et il ne s'est pas occupe qu'incidemment".[55] After this cautionary statement, he went on to state - possibly not without some embarassement - that Poisson had claimed that the force perpendicular to the plane of oscillation was too small to have an appreciable effect on the pendulum, and concluded somehow lamely:

> Cette conclusion parait contraire aux expériences de M. Foucault; mais le passage que je viens de citer permet un doute: Poisson ne rapporte pas le calcule de la force dont il parle, et d'ailleurs il n'est pas suffisant d'avoir reconnu qu'une force perturbatrice est très-petite pour conclure qu'elle ne produira qu'un effet insensible après un grand nombre d'oscillations.[56]

He then started an analysis of the problem in verbal form in which he made use of Poinsot's methods, considering the rotation of the pendulum as represented by a vector which could be decomposed into two parts, one of which was linked to fictive centrifugal forces that could be regarded as causing the pendulum to deviate.[57] One week later Binet complemented his first memoire with the relevant analytical formulas written in Poisson's notations, and recovered the desired sine factor.[58]

At the same time, Poinsot published a short note in which he offered no formulas, but a physical interpretation of the pendulum experiment in terms of the notions on which he had built his dynamics of rotation: he explained that it was misleading to regard the movement of the pendulum as due to some force because the phenomenon did not "fundamentally" ("au fond") depend on gravity or any other force.[59] The key feature of the pendulum, explained Poinsot, was not that its plane of oscillation moved, but that it remained constant, or rather attempted to remain as constant as possible under given conditions. It would be interesting, he continued, to construct a device whose plane of rotation would remain perfectly invariant with respect to "absolute space".[60] He described such an instrument, which involved an oscillating spring, and explained that, in this case, the "couple animating [the device] in the beginning" would be conserved.[61]

Thus, Poinsot interpreted Foucault's pendulum as a partial expression of the conservation of couples on which he had long since attracted attention, and proposed a new experiment demonstrating the conservation in perfect form. Foucault apparently did not build Poinsot's spring-contrievance, but he did construct an instrument which represented Poinsot's conservation of couples in the most perfect form: the gyroscope. Foucault realized this device one year after the pendulum, in 1852, and he did so by employing Poinsot's theory of rotation, and possibly also by discussing the problem with him in person.[62] The gyroscope, which had already been conceived by other authors, is an instrument which is build and set up in a frame in such a way, as to be able (at least ideally) to rotate free from the action of gravity and of friction. Under such ideal conditions, of which Foucault managed to give an extremely good approximation, the "couple" of the device remained constant in intensity and direction, and therefore the instrument could be seen to mantain always the same orientation with respect to the fixed stars. The idea of the gyroscope was not new, and other scholars and practitioners worked at building one, yet Foucault was the first one to present a working model to the Paris Academy and in 1854 he travelled to England and demonstrated the device at a meeting of the British Association for the Advancement of Science.[63]

Foucault's experiments had started an interest in rotations both in academic circles and among the

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[54] Binet 1851, p. 157.
[55] Binet 1851, p. 157.
[56] Binet 1851, p. 157-158.
[57] Binet 1851, p. 158.
[58] Binet 1851, p. 197-205.
[59] Poinsot 1851a, p. 206.
[60] Poinsot 1851a, p. 206.
[61] Poinsot 1851a, p. 207.
[62] Foucault 1852; Tobin 2003, p. 161.
[63] Tobin 2003, p. 166-167.

broader public and brought attention also to Poinsot's theory of rotation: as we have seen, the expanded version of his treatise on the subject was published in 1851, possibly in context of the enthusiasm for the pendulum, and a second printing came out a year later, as the gyroscope appeared.[64] However, the physical notions that Poinsot had associated to his formalism did not gain any followers in France and so, to follow the emergence of angular momentum, we shall have to move our attention to Britain.

9. The theory of couples in Great Britain and the definition of "angular mometum" by Robert Baldwin Hayward (1856)

In the same year in which the French original of Poinsot's "New theory of the rotation of bodies" (1834) was published, an English version of the work appeared under the title "Outlines of a new theory of rotatory motion".[65] The English translator had added a commentary and also appended to the booklet the translation of those passages or Poinsot's memoire from the year 1806 which dealt with the conservation of forces and moments. Poinsot's avoidance of analytical computations made his work particularly suitable for Brisish readers. An early reception of Poinsot's theory of rotation took place in Ireland, where a reform of mathematics had been started in 1813.[66] In 1844 James MacCullagh lectured at Dublin university on the theory of couples and also expanded on Poinsot's results.[67] He made use of analytical methods, but also took over the Frenchman's interpretation of rotational motion in terms of a conserved couple.[68] In 1845 and 1848 and William Rowan Hamilton presented to the Royal Irish Academy two papers in which he discussed the application of his method of quaternions to Poinsot's and MacCullagh's results.[69] The theory of couples also appeared in other works, as for example the "Mathematical principles of mechanical philosophy" (1836) by John Herny Pratt.[70] However, in these works no particular emphasis was put on the physical quantity which Poinsot had claimed to have discovered and which he had referred to as a conserved "force", "moment" or "couple" associated to the rotational motion of a body. Indeed, both MacCullagh and Hamilton followed rather an analytical than a geometrical approach. The first author to give prominence - and a new name - to Poinsot’s “conserved couple” was the mathematician Robert Baldwin Hayward.[71] Hayward had studied in London and Cambridge and had been 4th wrangler in the 1850 Tripos, thus being fully immersed in the Cambridge style of mathematical and physical education, which gave particular prominence to Newton’s geometrical approach to calculus and to the notion of force as impulse.[72] Hayward would later become a schoolmaster in mathematics, but in 1856 he was in Cambridge presenting to the Philosophical Society a paper on rotational motion in which he introduced “angular momentum”, discussing Foucault's pendulum as an example. [73] His paper started with two quotations by Poinsot on the necessity of going beyond analytical formulas to pursue science and continued: "My object is not so much to obtain new results, as to regard old one from a new point of view which renders all our equations directly significant."[74]

Hayward offered a treatment of the motion of a three-dimensional body which made use of analysis, but at the same time he refined and exploited the geometrical-physical notions introduced by Poinsot. The first part of the paper was purely mathematical, showing how to manipulate quantities

[64] Poinsot 1852.
[65] Poinsot 1834b.
[66] Grattan-Guinness 1990, p. 432-433.
[67] MacCullagh 1849, Moyer 1973.
[68] MacCullagh 1849, p. 335-336.
[69] Hamilton 1845, 1848.
[70] Pratt 1836, p. 20.
[71] Anon. 1950; Hayward 1856.
[72] Harman 1998, p. 19-27.
[73] Hayward 1856, p. 18-20.
[74] Hayward 1856, p. 1.

which we would call vectors and axial vectors.[75] At the beginning of the second part, the author wrote:

> [...] since every system of forces is reducible to a single force and a single couple, we have to investigate the effects of that force, and the effects of that couple. Now we know that the resultant force determines the motion of the centre of gravity of the system, be the constitution of the system what it may. In like manner the resultant couple determines something relatively to the motion of the system about its centre of gravity, which in the case of an invariable system defines its motion of rotation about that point, but which in other cases is not usually recognized as a definite objective magnitude, and has therefore no received name. This defect will be remedied by adopting momentum as the intermediate term between force and velocity, and by regarding as distinct steps the passage from force to momentum and that from momentum to velocity. In accordance with this idea we proceed to show that as in our first problem we shall be concerned with the magnitudes, force, linear momentum or momentum of translation, and linear velocity or velocity of translation, so in the other we shall be concerned with the corresponding magnitudes, couple, angular momentum or momentum of rotation, and angular velocity or velocity of rotations.[76]

Hayward interpreted Poinsot's theory by resolving what he perceived as a tension between velocity and force (i.e. between movement and its cause) by introducing the notion of momentum, and in particular of angular momentum. In this way he set a new, abstract representation of rotational movement which had emerged in analysis and had been geometrized by Poinsot on the same footing as the old idea of the "momentum", i.e. the "impulse" of a moving body. One may imagine that this step was made easier by the growing familiarity with spinning tops, gyroscopes, train wheels and engines offering a three-dimensional, dynamical representation of the force of rotation. Like Poinsot had done, Hayward gave particular prominence to the conservation of linear and angular momentum and underscored the continuity between the two notions by speaking of a "conservation of momentum" which could be applied both to the linear and the angular one, corresponding respectively to the "conservation of motion of the centre of gravity" and to the "principle of the conservation of areas".[77] Hayward remarked that some elements of his theory could be expressed in terms of Hamilton's quanternions.[78]

10. James Clerk Maxwell's spinning tops (1855-56)

Hayward's new formulation of the rotation of extended bodies was immediately noticed by a key figure of 19th century science: James Clerk Maxwell.[79] Maxwell had started his studies in his native Scotland, at the University of Edinburgh, and had continued them in Cambridge. In 1849 he had witnessed the experiments performed in Edinburgh by James David Forbes with spinning tops carrying discs painted in sectors of different colours with the aim of studying the composition of colours, and in 1854-55 he took up the same line of research.[80] In 1856, possibly after having experimented with the gyroscope, he published a short note "On an instrument to illustrate Poinsot's theory of rotation", where the instrument in question was none other than a spinning top carrying colored discs: "On the upper part of the axis [of the spinning top] is placed a disc of card, on which are drawn four concentric rings. Each ring is divided into four quadrants, which are coloured red, yellow, green, and blue. The spaces between the rings are white. When the top is in motion, it is easy to see in which quadrant the instantaneous axis is at any moment and the distance between it and the axis of the instrument."[81] Thus, Maxwell had interpreted a rotating instrument he was

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[75] Hayward 1856, p. 1-7. See also Caparrini 2002, p. 176-177. As Caparrini notes, in 1892 Hayward published a book on vector algebra.
[76] Hayward 1856, p. 7.
[77] Hayward 1856, p. 9.
[78] Hayward 1856, p. 12.
[79] My discussion of Maxwell's life and work are largely based on Harman 1998.
[80] Harman 1998, p. 37-48; Maxwell 1855.
[81] Maxwell 1856, p. 247.

familliar with as a representation of a geometrical-analytical theory of rotation, like the pendulum or the gyroscope.

One year later Maxwell published a much longer essay "On a dynamical top, for exhibiting the phenomena of the motion of a system of invariable form about a fixed point, with some suggestions as to the Earth's motion."[82] This time, the reference to instruments demonstrating rotational phenomena was very prominent: Maxwell started his paper stating that "To those who study the progress of science, the common spinning top is a symbol of the labours and the perplexities of men who had successfully threaded the mazes of planetary motions." and then went on to praise a series machines which had been used to visually represent the intricacies of rotation, among them the Earth model of Johann Bohnenberger and Foucault's gyroscope.[83] Before describing his spinning top, Maxwell expounded briefly the theory of rotation following the method of Poinsot, which he praised as "the only one which can lead to a true knowledge of the subject".[84] He then acknowledged the "important contribution" made by Hayward, giving the full reference of his paper, and then choosing as the centre of his treatment Hayward's notion of "angular momentum" and of its conservation "in direction and magnitude".[85] In his study of Maxwell's natural philosophy, Peter M. Harman remarks that Maxwell's appreciation of Poinsot's geometrical approach and of the notion of angular momentum can be understood in the context of the "Newtonian" tradition of a geometrical interpretation of calculus and of a mechanics based on the notion of "force" with which Maxwell had come into contact during his study in Edinburgh and Cambridge.[86] Maxwell made use of the notion of angular momentum and its conservation also in the essay on the stability of Saturn's rings written for the Adams prize of the University of Cambridge in 1857.[87] For our subjet it is important to remark that also in this case Maxwell built a mechanical instrument whose motion represented the dynamics he was discussing in analytical form.[88] As Harman noted "the abstractions of Cambridge mathematics were rendered visual, and transformed into Scottish physical realism."[89] I would like to underscore the fact that the contribution of mechanical models (spinning top, pendulum, gyroscope, Saturn's rings) were by no means a by-product of the knowledge-building process and instead contributed to shape it in an essential way. As we have seen, such devices did not just "visualize" theories, but rather represented a step along a complex path of physical-mathematical abstraction: they were conceived on the basis of refined analytical notions (e.g. Euler's equations) and complemented them by offering a representation of rotations which could be seen as fitting not only Poinsot's geometrical model, but also his natural philosophical interpretation of the dynamics of bodies based on a extension of the "Newtonian" notion of force. As to the "Scottish physical realism", it is intereseting to note that, in his essay on the rings of Saturn, Maxwell put the conservation of angular momentum on the same footing as the conservation of energy, and the same was done more or less at the same time by two other Scottish natural philosophers who most contributed to creating the "science of energy": William Thomson and William John Macquorn Rankine.[90]

11. William J. M. Rankine: angular momentum and applied mechanics (1858)

William John Macquorn Rankine had studied at the University of Edinburgh, but had left without taking a degree and had subsequently worked as an engineer, at first mostly in railway and train construction.[91] He had devoted much attention to rotations and in particular to the stress to which

[82] Maxwell 1857.
[83] Maxwell 1957, p. 248. On Bohnenberg's machine, a model of Earth precession see: Broelmann 2002, p. 37-41.
[84] Maxwell 1857, p. 250.
[85] Maxwell 1857, p. 250.
[86] Harman 1998, p. 13-27, 35-36.
[87] Harman 1998, p. 48-57, especially p. 55.
[88] Harman 1998, p. 58.
[89] Harman 1998, p. 57.
[90] Smith 1998.
[91] Hutchinson 1981; Parkinson 1975.

rotating elements such as railroad axles were subjected. Later on, he published extensively both on engineering and on the theory of matter and heat. Around 1850 he developed a theory of matter, heat and light based on the notion of "molecular vortices".[92] In these essays, no notion similar to angular momentum played an important role, but, as we shall see, they later became the basis for some reflections by William Thomson which are of relevance for the present subject.

In 1858 Rankine published a very influential "Manual of applied mechanics" in which he used both the name and the notion of angular momentum.[93] The book contained both well established results and recent innovation in the field and treated extensively all aspects of material stress and stability. The author set much worth in connecting theory and practice, and therefore at the beginning expounded the general principles that should be applied to the individual cases. Rankine introduced angular momentum when discussing systems of interacting bodies. He explained how to compute the absolute value of the quantity and then stated:

> Angular momenta are compounded and resolved like forces, each angular momentum being represented by a line whose length is proportional to the magnitude of the angular momentum and whose direction is perpendicular to the plane of the motion of the body and of the fixed point and such, that when the motion of the body is viewed form the extremity of the line, the radius vector of the body seems to have a right-handed rotation.[94]

This definition took care of all possible ambiguities. Rankine demonstrated the conservation of angular momentum for a system of mass points and stated that this law was sometimes called the "pinciple of the conservation of areas".[95] In the first edition of the manual, Rankine referred to the work on rotation by Poinsot and Maxwell, but he did not mention Hayward.[96] In later editions of the work, however, he acknowledged that "The term angular momentum was introduced by Mr. Hayward".[97]

Later on, he discussed the motion of rigid bodies and right at the beginning stated that the variations of linear momentum were due to the resultant external force, while those of angular momentum were the effect of the resultant couple.[98] After having defined angular momentum for a solid body, Rankine stated that also in this case the conservation law was valid and took this principle together with the conservation of energy as a starting point for his discussion of the motion of a free rotating body.[99]

12. William Thomson's "momentum of momenta" and the magnetic properties of matter (1857)

We now turn to a third representative of the "Scottish physical realism": William Thomson (from 1897 Lord Kelvin). Thomson had learned about Poinsot's theory of couples already in 1839, when he was only fifteen years old, studying at Glasgow college. His teacher John Pringle Nichols, who also introduced him to the work of Jean Baptiste Joseph Fourier on heat transmission, had "recently got hold of a new book – a pamphlet of some eighty pages – on Couples, and made his students write Christmas essays on the Theory of Couples".[100] The pamphlet was either the English translation of Poinsot's book or the French original. In 1840 Thomson bought himself also a copy of another memoire by Poinsot which dealt with the equilibrium conditions.[101] In 1845, when he

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[92] Rankine 1851a 1851b.
[93] Rankine 1858.
[94] Rankine 1858, p. 505.
[95] Rankine 1858, p. 506-507.
[96] Rankine 1858, p. 535.
[97] For example in the fourth edition: Rankine 1868, p. 506.
[98] Rankine 1858, p. 513.
[99] Rankine 1858, p. 529-534.
[100] Thomson S. P. 1910, p. 13, 73.
[101] Smith and Wise 1989, p. 366.

was at the University of Cambridge, Thomson spent some time both experimenting with rotating bodies and reflecting on the mathematics of rotation.[102] In the following years he did not study the subject further, but in the 1850's he took an interest in the theory of "molecular vortices" which, as already mentioned, Rankine had developed to explain heat phenomena.[103] While Rankine had made no use of the notion of angular momentum, in Thomson's theory it played a key role to bridge the gap between mechanics and electromagnetism.

Rankine had proposed a quite detailed mathematical theory of matter according to which the elements of matter had a more or less spherical form and were constituted by a nucleus and a fluid atmosphere. The fluid in the atmosphere moved in vortices having their axes of rotation directed along the radii of the sphere. It is not necessary for us to go into the details of Rankine's model, but only to note that in 1857 Thomson took it as a starting point to offer a "Dynamical illustration of the magnetic and the helicoidal rotatory effect of transparent bodies on polarized light".[104] In his paper Thomson offered a mechanical explanation of the effect of magnetism on the transmission of polarized light through a transparent medium. Thomson proposed to consider the velocitiy of transmission of light as resulting from the composition of the velocity of the light wave with that of rotational motions internal to the body, such as Rankine's molcular vortices. Thomson recalled that Ampère had already linked magnetism to microscopical circulating electrical currents and stated:

> Hence it appears that Faraday's optical discovery [i.e. the effect of magnetism on light] affords a demonstration of the reality of Ampère's explanation of the ultimate nature of magnetism; and gives a definition of magnetization in the dynamic theory of heat. The introduction of the principle of moments of momenta ("the conservation of areas") into the mechanical treatment of Mr. Rankine's hypothesis of "molecular vortices," appears to indicate a line perpendicular to the plane of the resultant rotatory momentum ("the invariable plane") of the thermal motions as the magnetic axis of a magnetized body, and suggests the resultant moment of momenta of these motions as the definite measure of the "magnetic moment".[105]

As we see, Thomson here made use of the notion of angular momentum ("moment of momenta") and of its conservation, for which he quoted in brackets the traditional analytical names, probably for the benefit of some readers. He proposed to identify the "moment of momenta" of the vortical motions with magnetic moment: an idea which survived not only his model, but also classical mechanics and electromagnetism, to be taken over into quantum theory. Thomson offered no mathematical details of how the theory should look like, in contrast to Rankine, who had developed a very detailed hydrodynamical model for the vortices. On the contrary, Thomson professed himself completely agnostic as to the exact mechanism of matter:

> The explanation of all phenomena of electromagnetic attraction and repulsion, and of electromagnetic induction, is to be looked for simply in the inertia and pressure of the matter of which the motions constitute heat. Whether this matter is or is not electricity, whether it is a continuous fluid interpermeating the space between molecular nuclei, or is itself molecularly grouped; or whether all matter is continuous, and molecular heterogeneousness consists in finite vortical ot other relative motions of contiguous parts of a body, it is impossible to decide, and perhaps in vain to speculate, in the present state of science.[106]

The notion of moment of momentum was particularly fitting to Thomson's attitude: on the one side it was a rigorously defined mathematical-mechanical notion, while on the other it did not require detailed speculations on the mechanical structure of matter.[107] The connection to the magnetic moment appear plausible because that quantity, too, was usually represented by means of an oriented segment and, since angular momentum was known to be conserved, the link could be regarded as valid independently of the continuous movements going on inside matter. Thomson's theory later provided a starting point for Maxwell's electromagnetism and, although the hypothesis

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[102] Thomson S. P. 1910, p. 124, 737.
[103] Rankine 1851a, 1851b; Thomson 1857.
[104] Thomson 1857.
[105] Thomson 1857, p. 152.
[106] Thomson 1857, p. 152.
[107] Harman 1982, p. 69-71.

of molecular vortices would eventually be discarded, the connection between magnetic moment and angular momentum remained.[108] Thus, Thomson had taken up the idea that behind the conservation of moment of momenta lay a physical quantity of particular relevance and had connected it with a phenomenon of non-mechanical nature: magnetic moment.

In the 1860s Thomson teamed up with another Scottish natural philosopher, Peter Guthrie Tait, to write a "Treatise on natural philosophy" which should offer an overview of that discipline in which mathematics would closely fit physics.[109] Most prominent among their principles of natural philosophy was the conservation of energy, but Thomson and Tait also made large use of simple machines such as the screw to express the contents of their subject, and supported the geometrical-physical formalisation of mechanics which underscored the significance of vectorial notions like "momentum" and "momentum of momentum".[110] Because of Thomson's oppositions, the book made no use of Hamilton's quaternions, even though Tait was "an ardent disciple of Hamilton", as Maxwell put it, regretting that the manual did not employ that new analytical tool.[111] Once again, we see how the choice of mathematical forms was closely linked to personal images of scientific knowledge: Thomson saw quaternions and vector algebra as a hindrance to physical understanding, rather than as a formalisation which gave prominence to physical meaning, as modern physicists regard it. Following Rankine's example Thomson and Tait stressed the analogy between linear and angular momentum, stating their conservation laws and adding at the end that the conservation of momentum of momentum "is sometimes called Conservation of areas, a very misleading designation"[112]

13. Angular momentum at the crossroad between geometry, natural philosophy and engineering

In the previous sections I have endeavoured to show how the notion of angular momentum emerged from the convergence of a number of factors: the development of the mechanical analysis of rotational motion by French mathematicians; the reinterpretation and expansion of these results in new physical-geometrical terms; some specific natural philosophical ideas of motion and its causes and, finally, the construction, use and discussion of various mechanical instruments representing the properties of rotational motion. Some crucial steps in this process were taken in Britain, where both geometrical formalism and mechanical models were more present in the academical milieu than in other European countries and enjoyed a higher epistemological status. In the context of Victorian natural philosophy the notion of angular momentum could emerge and thrive because it was supported by different but complementary representations of nature and its regularities.

The example of Thomson's theory of magnetism and molecular vortices has shown how angular momentum, being linked not only to a specific mathematical formalism, but also to a physical picture, could provide a means of exporting analytical mechanical ideas and methods into other areas of science, as was also the case in Maxwell's mathematisation of electromagnetic theory.[113]

During the second half of the 19th century rotating machines of various kinds were used by British scientists not only to demonstrate theoretical models of natural phenomena (atomic structure, heat theory, electromagnetism), but also to translate them into a new formalism which eventually allowed to develop them further, as in the case of Tait's "smoke ring" demonstration of Hermann Helmholtz's theory of hydrodynamic vortices or Thomson's frequent use of gyrostats to model electromagnetic theories .[114]

Outside of Britain, however, the notion of angular momentum did not have much fortune. In France

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[108] Harman 1998, p. 109-112, 115-124.

[109] Thomson and Tait 1867. On the book see: Smith and Wise 1989, p. 348-395.

[110] Thomson and Tait 1867, p. 173-187; Smith and Wise 1989, p. 365-372.

[111] Smith and Wise 1989, p. 365-366.

[112] Thomson and Tait 1867, p. 187.

[113] Harman 1998, p. 98-112.

[114] Broelmann 2002, p. 77-80; Silliman 1963, especially p. 46; Smith and Wise 1989, p. 438-439, 473-475, 485-488.

Jean Marie Constant Duhamel devoted much space in his textbook of mechanics to the theory of couples, but only mentioned as an aside the fact that the moment of the quantitiy of motion was conserved in absence of external forces and moments of forces, and presented this result as an "application of the principle of areas".[115] In Germany Hermann von Helmholtz formulated a refined theory of vortex motion in matter, in which however the notion of angular momentum did not appear, although Helmholtz made use of Poinsot's formalism to compose rotation with the parallelogram rule.[116] Ernst Mach, in his treatise on "Die Mechanik in ihrer Entwicklung" (1883), explained the law of "conservation of areas" without mentioning angular momentum and only added at the end of the discussion that this was "a generalization of the principle of inertia”.[117]

14. The "Theory of the spinning top" by Felix Klein and Arnold Sommerfeld (1897-1903)

The first German text in which angular momentum was presented as a physical quantity of relevance was the treatise "Über die Theorie des Kreisels" (1897-1910) written by Felix Klein together with Arnold Sommerfeld.[118] The book was due to the initiative of Klein, who had been pursueing the aim of reintroducing geometrical methods into mathematics, and it was an innovative attempt to combine the analytical and the geometrical approach to the study of rotation. It is no chance that the text put at its centre a mechanical device, the spinning top, as representation of rotational motion, since the authors repeatedly quoted and praised Thomson and Tait, and followed them in making use of a "Newtonian" concept of force.[119] They also acknowledged their debt to Poinsot, whose "beautiful methods" ("schöne Methoden") they cultivated in their treatise, and giving particular importance to a notion of impulse:

> Noch wichtiger für uns aber ist die volle Klarheit über die mechanischen Ursachen der Bewegung , über die ins Spiel kommenden Kräfte. Wir werden uns diese möglichst konkret im Raume als Vektoren versinnlichen; besonders Wert legen wir auf die Ausbildung und konsequente Benutzung des Impulsbegriffs, worunter wir diejenige Stosskraft verstehen, welche imstande ist, die jeweilige Bewegung momentan von der Ruhe aus zu erzeugen.[120]

While one might be tempted to equate the "impulse" with linear momentum, this was only true for point masses: in the case of solid bodies, the impulse was divided into a translational and a rotational part, which Klein and Sommerfeld in the first volume of the work (1897) referred to as "Schiebeimpuls" and "Schraubeimpuls", while in later volumes the term "Drehmoment" was introduced.[121] The authors made clear that their notion of rotational impulse was precisely the one introduced by Poinsot: "Der Begriff des Impulses des Kreisels ist von Poinsot in den mehrfach zitierten Arbeiten vollständig entwickelt worden. Die Bezeichnung Poinsot's lautet etwas umständlich couple d'impulsion."[122] I would like to suggest that this emphasis on angular momentum as a quantity as physically foundamental as linear momentum may have played a role a few years later, when Sommerfeld tackled the problem of the quantization of atomic motion. In the last part of this paper I shall briefly discuss how the notion of angular momentum was used as a means to bridge the gap between classical and quantum physics.

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[115] Duhamel 1863, p. 178-179.
[116] Helmholtz 1858. On Helmholtz theory of hydrodynamic and magnetic vortices see: Silliman 1963, p. 462-463.
[117] Mach 1883, p. 171-173, 281.
[118] Klein and Sommerfeld 1897-1903.
[119] Harman 1982, p. 69-70; Klein and Sommerfeld, 1897, p. 69.
[120] Klein and Sommerfeld 1897, p. 4-5.
[121] Klein and Sommerfeld 1897, p. 70-104; Klein and Sommerfeld 1903, p. 514.
[122] Klein and Sommerfeld 1897, p. 104.

15. Angular momentum and the quantum: Niels Bohr's atomic model (1913)

In a series of studies published in 1913 Niels Bohr proposed his highly innovative atomic theory.[123] Starting point for his reflections was Ernest Rutherford's model of the atom as a microscopic Solar system with an electron orbiting around a positively charge nucleus. This motion, when treated according to classical mechanics and electromagnetism, was known to give rise to unstable configurations in which the atom would steadily lose energy through radiation and eventually collapse. Bohr's crucial step was to assume the existence of "stationary states" in which atoms did not radiate and therefore maintained a constant value of the energy. Bohr computed the stationary energy values by first making use of classical formulas and then imposing on the result an additional condition involving Planck's constant h and an integer number τ (“quantum number”). The condition was such, that agreement with observed data could be obtained and, for the hydrogen atom, the energy W was bound to have the form:
$W(\tau) = (2\pi^2 m e^4)/(h^2 \tau^2)$. [124]
Here m and e were respectively the mass and charge of the electron. Radiation took place in separate emissions or absorptions associated to the transition of the atom from one stationary state to another. Bohr could not offer any formal description of these "jumps" other than the frequency condition $W_{initial}-W_{final}= h\nu$, where ν was the frequency of the emitted light.[125]
Bohr's theory could predict the values of the spectral lines of hydrogen and also qualitatively explain the discrete structure of atomic and molecular spectra. Yet he recognized that his model, while successful from the phenomenological point of view, hardly provided a physical explanation for atomic structure, and offered a tentative interpretation of his results in terms of what he called "symbols taken from ordinary mechanics".[126] He pointed out that the quantization condition for the energy took a very simple form when expressed in terms of angular momentum: "If we therefore assume that the orbit of the electron in the stationary states is circular, the result of the calculation on p. 5 [i.e. the formula for W] can be expressed by the simple condition: that the angular momentum of the electron round the nucleus in a stationary state of the system is equal to an entire multiple of a universal value, independent of the charge on the nucleus."[127]
So Bohr obtained for the angular momentum M the condition: $M= \tau\, h/(2\pi)$, where τ was again an integer quantum number. This expression had the same form of the various quantization conditions that, following the success of Max Planck's black body radiation formula, had been employed in various fields of physics.[128] Bohr's condition corresponded to quantizing the absolute value of angular momentum and his specification that one should assume circular orbits indicates that he was not making any effort to give a detailed physical interpretation of his model: the “self-evident” connection between the physical notion of angular momentum and quantum physics was all but evident to him.

16. Arnold Sommerfeld's atomic angular momentum and its connection to magnetic moment (1915-1919)

While Bohr had regarded the analogy between classical angular momentum and the quantity involved in his atomic model as purely symbolical, Arnold Sommerfeld took the opposite stance. In a series of papers published from 1915 onward he expanded and refined Bohr's theory in such a way as to accomodate three integer quantum numbers instead of only one, and was able to at least partly make sense of the fine structure of atomic spectra as well as of the characteristic of the radiation

[123] For the present study, it sufficec to discuss the contents of Bohr's first article Bohr 1913. For a discussion of the early stages of development of quantum theory see for example Jammer 1966, p. 69-88.
[124] Bohr 1913, p. 8.
[125] Bohr 1913, p. 8.
[126] Bohr 1913, p. 15.
[127] Bohr 1913, p. 15.
[128] Bohr 1913, p. 15; Jammer 1966, p. 46-61.

emitted and absorbed under the influence of magnetic or electric fields.[129] What is of particular interest for us is that Sommerfeld achieved his results not purely on the basis of analytic prowess, but also by following a physical picture of stationary states in which the "mechanical" notion of angular momentum served as a means to bridge the gap between classical and quantum theory.

Like Bohr, Sommerfeld considered the atom as a Keplerian system in which a small electron orbited around a large nucleus, but other than Bohr he deployed the whole apparatus of analytical mechanics to consider the motion of the system, expressing it in terms of the canonical conjugate variables (q, p).[130] The position q was expressed at first in polar coordinates (r, θ), later in spherical ones (r, θ, ψ), and in both cases the generalized momenta p corresponded to angular momentum. In the first case Sommerfeld only took into account the two degrees of freedom of the electron on the plane of the (elliptical) orbit, whose dimensions and eccentricity could vary, and so the angular momentum p was constrained to be in the direction perpendicular to the orbit and only had a single degree of freedom. Thus, even when allowing elliptical orbits, imposing quantization conditions on this classical constellation amounted to quantize only the absolute value of angular momentum, like Bohr had done and, unsurprisingly, Sommerfeld in the end obtained exactly the same result as Bohr had reached. To go beyond, he decided to quantize all three degrees of freedom of p, which amounted to quantizing not only the dimensions and eccentricity of the orbit, but also its orientation in 3-dimensional space - a "space quantization" ("Raumquantisierung"), as it would be called later.[131] This step proved essential for taking into account relativistic effects and so finally going beyond Bohr's model and explaining how fine structure of hydrogen and the multiplet structure of complex spectra depended on two quantum numbers.[132]

Despite the phenomenological success of his model, Sommerfeld felt that space quantisation required some physical justification, since it implied an arbitrary choice of a preferred direction in space - the z-axis of spherical coordinates - to be used when imposing physically relevant quantization conditions. Therefore, he introduced the procedure with these remarks:

> Es entsteht die Frage, ob sich auch die Lage der Bahn "quanteln" läßt. Dazu muß allerdings wenigstens eine Bezugsebene im Raum ausgezeichnet sein, sei es durch ein äußeres elektrisches oder magnetisches Feld oder durch die Konstitution des Kernes selbst, z.B. einen diesen umgebenden Elektronenring. Bei den kräftefreien Wasserstoffkern dagegen ist die Lage der Bahnebene aus Mangel an allen Bezugsstücken physikalisch unbestimmt und daher auch nicht quantentheoretisch bestimmbar. Wenn wir trotzdem eine Quantenbedingung für die räumliche Lage der Bahn am Wasserstoffmodell entwickeln werden, so ist dies folgendermassen gemeint: Wir denken uns durch einen (äußere oder innere) physikalische Ursache eine Richtung im Raum ausgezeichnet, lassen aber die Stärke derselben zu Null abnehmen, so daß wir wieder genau diese Ursache quantitativen Verhältnisse haben wie bei der Bewegung im Felde des reinen Wasserstoffkernes, aber mit der Möglichkeit der Orientierung gegen eine Vorzugsrichtung (oder Vorzugsebene). Diese Richtung können wir dann zur Achse, diese Ebene zur Äquatorebene eines räumlichen Polarkoordinatensystems r, θ, ψ wählen.[133]

Thus, Sommerfeld justified his apparently arbitrary choice of reference frame by imagining that a "physical cause" like a magnetic or electric field, if present, would constrain the motion of the system. He then let the intensity of the imaginary field go to zero, to obtain a preferred direction in space despite the rotational symmetry of the system. Apart from the obvious methodological problems inherent in this kind of "symmetry breaking", what is interesting for us is that here Sommerfeld was assuming that the "quantistic" angular momentum would be affected by electric and magnetic fields like its classical counterpart. In other words, he was implying that the mathematical formulas which were called "angular momentum" in his quantum theory stood in a

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[129] Jammer 1966, p. 89-96. In the following, I shall discuss Sommerfeld's results as presented in the paper: Sommerfeld 1916.

[130] Sommerfeld 1916, p. 14-28.

[131] Sommerfeld 1916, p. 28-33.

[132] Sommerfeld 1916, p. 44-94.

[133] Sommerfeld 1916, p. 29.

physical relation, and not just in purely symbolic analogy, to the classical quantity: like classical angular momentum, also the quantistic one determined the behaviour of the atom in an electromagnetic field. Without any support from experiments - which on the contrary suggested that classical theory did not apply to atoms - Sommerfeld was here postulating the validity of the same connection between rotation and magnetisation which had been proposed decades earlier by William Thomson. On the basis of this assumption he interpreted the quantum numbers linked to space quantization as establishing the number of possible orientations which "angular momentum" could take with respect to an external magnetic or electric field. For a quantum number n=1 two orientations were possible, for n=2 five, and so on.[134]

In his paper Sommerfeld did not commit himself explicity on whether atomic angular momenta could be considered equivalent to macroscopic ones, but in his textbook "Atombau und Spektrallinien" (1919) he clearly stated his opinion that, like energy, also linear and angular momenta, i.e. "Impuls" and "Impulsmoment" were to be understood as physical quantities whose properties could be expressed both in classical and in quantum terms. In 1918 Wojciech (Adalbert) Rubinowicz, who had formerly been an assistant to Sommerfeld in Munich, had postulated the conservation of “angular momentum” during the interaction between atoms and radiations and had used it to explain some selection rules of atomic spectra.[135] In his textbook Sommerfeld summarized and expanded this idea, giving his full support to Rubinowicz's results: "Wenn bei der Konfigurationsänderung des Atoms sich sein Impuls oder Impulsmoment ändert, so sollen sich diese völlig und ungeschwächt wiederfinden in dem Impuls und dem Impulsmomente der Strahlung."[136]

To appreciate the radicality of this statement one has to keep in mind that, at that time, no mathematical formalism for quantum theory existed in whose context the conservation could be formulated, let alone proven. Moreover, in 1918 Bohr had proposed an explanation of selection rules which did not require any physical interpretation of quantum numbers and was in better agreement with experiment than Rubinowicz's proposal. Bohr explicitely cast doubts on the conservation of angular momentum for quantum systems. However, Sommerfeld's physical interpretation of atomic angular momentum was vindicated against Bohr's skepticism by the experiment performed in 1921-22 by Otto Stern and Walther Gerlach.

17. The experiment of Otto Stern and Walther Gerlach: the operationalisation of quantum angular momentum (1921-22)

In the summer of 1921 Otto Stern wrote a paper proposing "a method to test experimentally the quantization of direction in a magnetic field"[137] Stern took Sommerfeld's idea on atomic angular momentum and its connection to magnetic moment at face value and suggested how they could be put to the test:

> In der Quantentheorie des Magnetismus und des Zeemaneffekts wird angenommen, daß der Vektor des Impulsmomentes eines Atoms nur ganz bestimmte diskrete Winkel mit der Richtung der magnetischen Feldstärke H bilden kann, derart, daß die Komponente des Impulsmomentes in Richtung von H ein ganzzahliges Vielfaches von $h/2\pi$ ist. Bringen wir also ein Gas aus Atomen, bei denen das gesamte Impulsmoment pro Atom - die vektorielle Summe der Impulsmomente sämtlicher Elektronen des Atoms - den Betrag $h/2\pi$ hat, in ein Magnetfeld, so sind nach dieser Theorie für jedes Atom nur zwei diskrete Lagen möglich, da die Komponente des Impulsmomentes in Richtung von H nur die beiden Werte $\pm h/2\pi$ annehmen kann.[138]

On the basis of Sommerfeld’s theory Stern treated the quantized angular momentum as a vector

[134] Sommerfeld 1916, p. 32-33, Sommerfeld 1919, p. 411-415. For a detailed discussion of this issue see: Weinert 1995, p. 80-83.
[135] On the explanation of selection rules in the old quantum theory see: Borrelli 2009.
[136] Sommerfeld 1919, p. 381.
[137] Stern 1921. On the Stern-Gerlach experiment and its significance see: Weinert 1995.
[138] Stern 1921, p. 249.

quantity J which really existed in space and had a given length, but an as yet undertermined orientation. He further assumed that this vector J was associated to a magnetic moment M = 1/2 e/m J, just like in the classical case.[139] When a beam of atoms passed through a magnetic field, their angular momentum was forced to orient itself with respect to the direction of the field in one of the two positions which were allowed by the quantum theory, and this would lead to a splitting of the beam into two parts. In the classical case, instead, the beam would simply spread in a continuous way and so it would be in principle possible to distinguish the two cases. A short time later, helped by Walther Gerlach, Stern performed the experiment with a beam of silver atoms which they assumed to correspond to the case n=1 and therefore to fulfil the conditions described by Stern in his theoretical paper.[140] The beam of atoms split into two parts and thus the authors could announce "that the quantisation of direction [of angular momentum] had proved to be a fact" ("Die Richtungsquantelung im Magnetfeld [wurde] als Tatsache erwiesen").[141]

The result was received with some astonishment by the scientific community, as few had actually regarded space quantization as more than a formal device, yet the discrete splitting of the beam offered an impressive evidence of the failure of classical theory and implicitly supported the belief that the quantum formalism for angular momentum indeed represented a physical quantity which was, if not identical, at least very similar to the classical notion bearing the same name. As in the case of Foucault's pendulum, two different representation of what was assumed to be a law of nature had been put near each other, and a new physical notion had emerged from that tension: quantum angular momentum. The fact that atomic angular momentum could be linked to magnetic moment - and vice versa - proved to be of the utmost importance for the later development of quantum theory, because it provided a means to operationalize and investigate the otherwise very abstract notion of atomic angular momentum: studying the behaviour of atoms in magnetic fields (Zeeman, Paschen-Bach effect). Eventually, this led to the emergence of the concept of spin and to the relativistic and quantum-field-theoretical generalisation of angular momentum.

18. Conclusions

Combining the mathematical analysis of motion with the geometrical representation of mechanical entities and with a Newtonian notion of "force", Louis Poinsot developed the physical-mathematical concept of a "conserved moment" which he used to further explore the dynamics of rotation. Around 1850 this idea proved capable of bridging the gap between analytical mechanics and the mechanical devices representing the properties of rotational motion (Foucault's pendulum, the gyroscope). In the context of the British and especially Scottish natural philosophy of the Victorian era, where geometrical reasoning and mechanical models had come to be regarded as having a particularly high epistemological value, Robert B. Hayward formulated the modern definition of "angular momentum", which was promptly taken up by James C. Maxwell and William J. M. Rankine and employed by William Thomson to establish a connection between the structure of matter and its electromagnetic properties. Later on, guided by the classical notion of angular momentum, Arnold Sommerfeld constructed its equivalent in quantum theory, and Otto Stern and Walther Gerlach established an operational definition for it which not only survived the old quantum theory, but eventually became central to quantum mechanics and quantum field theory.

I believe this picture offers an example of how physical-mathematical notions emerge and are constantly supported by the interactions and unresolved tensions between different representations of phenomena, of mathematical structures and of philosophical ideas in words, symbols, graphics, mechanical contrievances or by any other means. Thanks to this multiplicity, the actors making use of the notions can often find one aspect of the composite which fits the present needs, bridging the gap between different phenomena to be intrepreted, or different conceptualization of natural laws to be connected with each other.

[139] Stern 1921, p. 250-251.
[140] Gerlach and Stern 1921 and 1922.
[141] Gerlach and Stern 1922, p. 349.

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